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801 Homework 4

October 15, 2015

Problem 1:

(a) Show that the F test developed in the first part of this section is equivalent to the (generalized) likelihood ratio test for the reduced versus full models. (b) Find an F test for $H_0\colon X\beta = X\beta_0$ where β_0 is known. (c) Construct a full versus reduced model test when σ^2 has a known value σ_0^2.

Solution: (a) Let the full model be $Y = X\beta + \epsilon$ and the reduced model be $Y = X_0\gamma + \epsilon$, where $\epsilon \sim N(0, \sigma^2 I)$. Denote the likelihood under the full model L_F and the likelihood under the reduced model L_R. Then, the likelihood ratio is

$$r = \frac{\sup L_F(\sigma^2, \beta)}{\sup L_R(\sigma^2, \gamma)} = \frac{(\widehat{\sigma}_F^2)^{-n/2} \exp\{-(Y - X\widehat{\beta})'(Y - X\widehat{\beta})/2\widehat{\sigma}_F^2\}}{(\widehat{\sigma}_R^2)^{-n/2} \exp\{-(Y - X_0\widehat{\gamma})'(Y - X_0\widehat{\gamma})/2\widehat{\sigma}_R^2\}}.$$

First, note that the estimates for σ^2 under the full and reduced model is the MLE under those models, i.e.

$$\widehat{\sigma}_F^2 = \frac{Y'(I - M)Y}{n} \quad \text{and} \quad \widehat{\sigma}_R^2 = \frac{Y'(I - M_0)Y}{n}.$$

Then, we see that we can rewrite the exponentials as

$$\begin{aligned}
\exp\{-(Y - X_0\widehat{\gamma})'(Y - X_0\widehat{\gamma})/2\widehat{\sigma}_R^2\} &= \exp\left\{ -\frac{n}{2} \cdot \frac{(Y - M_0Y)'(Y - M_0Y)}{Y'(I - M_0)Y} \right\} \\
&= \exp\left\{ -\frac{n}{2} \cdot \frac{Y'Y - Y'M_0Y}{Y'Y - Y'M_0Y} \right\} \\
&= \exp\left\{ -\frac{n}{2} \right\}
\end{aligned}$$

and

$$\exp\{-(Y - X\widehat{\beta})'(Y - X\widehat{\beta})/2\widehat{\sigma}_F^2\} = \exp\left\{ -\frac{n}{2} \right\}.$$

Therefore, the ratio becomes

$$\begin{aligned}
r = \left(\frac{\widehat{\sigma}_F^2}{\widehat{\sigma}_R^2} \right)^{-n/2} &= \left(\frac{Y'(I - M)Y}{Y'(I - M_0)Y} \right)^{-n/2} = \left(\frac{Y'(I - M_0)Y}{Y'(I - M)Y} \right)^{n/2} \\
&= \left(\frac{Y'(I - M)Y + Y'(M - M_0)Y}{Y'(I - M)Y} \right)^{n/2} \\
&= \left(1 + \frac{r(M - M_0)}{r(I - M)} \cdot \frac{Y'(M - M_0)Y/r(M - M_0)}{Y'(I - M)Y/r(I - M)} \right)^{n/2}.
\end{aligned}$$

Since this is monotone increasing in $\frac{Y'(M-M_0)Y/r(M-M_0)}{Y'(I-M)Y/r(I-M)}$ and the F statistic is

$$F = \frac{Y'(M-M_0)Y/r(M-M_0)}{Y'(I-M)Y/r(I-M)},$$

we see the two test tests are equivalent.
(b) Let the full model be $Y = X\beta+\epsilon$. Then, the reduced model induced by H_0 is $Y^\star = X_0\gamma+\epsilon$, where $Y^\star = Y - X\beta_0$ and $X_0 = 0$. Then, note that $M = X(X'X)^-X'$ and $M_0 = 0$, and so $M - M_0 = M$. Then, we calculate the test statistic

$$\begin{aligned} F &= \frac{Y^\star(M-M_0)Y^\star/r(M-M_0)}{Y'(I-M)Y/r(I-M)} \\ &= \frac{(Y-X\beta_0)'M(Y-X\beta_0)/r(M)}{Y'(I-M)Y/r(I-M)}. \end{aligned}$$

Then, reject H_0 if $F > f\big(1-\alpha, r(M), r(I-M)\big)$.
(c) Recall from section 2.6 that

$$\frac{Y'(I-M)Y}{\sigma^2} \sim \chi^2\big(r(I-M)\big).$$

Then, under H_0, we calculate the test statistic

$$\chi_0^2 = \frac{Y'(I-M)Y}{\sigma_0^2}.$$

Therefore, reject H_0 if $\chi_0^2 < \chi^2\big(\alpha, r(I-M)\big)$ or if $\chi_0^2 > \chi^2\big(1-\alpha, r(I-M)\big)$.

Problem 2:

Redo the tests in Exercise 2.2 using the theory of Section 3.2. Write down the models and explain the procedure. **Exercise 2.2:** Let $y_{11}, y_{12}, ..., y_{1r}$ be $N(\mu_1, \sigma^2)$ and $y_{21}, y_{22}, ..., y_{2s}$ be $N(\mu_2, \sigma^2)$ with all y_{ij}'s independent. Write this as a linear model. Find estimates of $\mu_1, \mu_2, \mu_1 - \mu_2$, and σ^2. Form an $\alpha = .01$ test for $H_0\colon \mu_1 = \mu_2$. Similarly, form 95% confidence intervals for $\mu_1 - \mu_2$ and μ_1. What is the test for $H_0\colon \mu_1 = \mu_2 + \Delta$, where Δ is some known fixed quantity? How do these results compare with the usual analysis for two independent samples?

Solution: The full linear model can be written as $Y = X\beta + \epsilon$, where

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1r} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2s} \end{bmatrix},$$

and $\epsilon \sim N(0, \sigma^2 I)$. We want to form an $\alpha = .01$ test for $H_0\colon \mu_1 = \mu_2$, i.e. the reduced model is $Y = X_0\gamma + \epsilon$, where

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1r} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2s} \end{bmatrix},$$

and $\epsilon \sim N(0, \sigma^2 I)$. We have the projection matrices

$$M = X(X'X)^{-}X' = \begin{bmatrix} \mathbf{\frac{1}{r}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\frac{1}{s}} \end{bmatrix}_{(r+s)\times(r+s)}$$

and

$$M_0 = X_0(X_0'X_0)^{-}X_0' = \begin{bmatrix} \mathbf{\frac{1}{(r+s)}} \end{bmatrix}_{(r+s)\times(r+s)}.$$

Then, we obtain the test statistic to be

$$\begin{aligned} F &= \frac{Y'(M-M_0)Y/r(M-M_0)}{Y'(I-M)Y/r(I-M)} \\ &= \frac{(MY)'(MY) - (M_0Y)'(M_0Y)/r(M-M_0)}{Y'Y - (MY)'(MY)/r(I-M)} \\ &= \frac{(X\widehat{\beta})'(X\widehat{\beta}) - (X_0\widehat{\gamma})'(X_0\widehat{\gamma})/r(M-M_0)}{Y'Y - (X\widehat{\beta})'(X\widehat{\beta})/r(I-M)}. \end{aligned}$$

Recall from homework 3 problem 1 that

$$\widehat{\beta} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \end{bmatrix} \quad \text{and} \quad \widehat{\gamma} = \begin{bmatrix} \overline{y}_{12} \end{bmatrix}$$

where $\overline{y}_{12} = \frac{1}{r+s}(y_{11} + ... + y_{2s})$. Multiplying these by X and X_0, respectively, gives

$$F = \frac{\left(r\overline{y}_1^2 + s\overline{y}_2^2 - (r+s)\overline{y}_{12}^2\right)/1}{\left(\sum_{i=1}^{r}(y_{1i}^2 - \overline{y}_1^2) + \sum_{i=1}^{s}(y_{2i}^2 - \overline{y}_2^2)\right)/(r+s-2)}.$$

Then, reject H_0 if $F > f(1-\alpha, 1, r+s-2)$.
Now we wish to find the test for $H_0\colon \mu_1 = \mu_2 + \Delta$. Let the full model be $Y = X\beta + \epsilon$, where $\epsilon \sim N(0, \sigma^2 I)$, i.e.

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1r} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2s} \end{bmatrix}.$$

The reduced model $Y = X_0\gamma + \epsilon$ can be written as

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_2 + \Delta \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1r} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2s} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1r} \\ y_{21} \\ \vdots \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_2 \end{bmatrix} + X \begin{bmatrix} \Delta \\ 0 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1r} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2s} \end{bmatrix}.$$

Therefore, defining $Y^\star = Y - Xb$, where $b = (\Delta, 0)'$, we have the reduced model $Y^\star = X_0\gamma + \epsilon$, where $X_0 = J_{r+s}, \gamma = \begin{bmatrix} \mu_2 \end{bmatrix}$, and $\epsilon \sim N(0, \sigma^2 I)$. Then, we calculate the test statistic

$$\begin{aligned} F &= \frac{Y^{\star\prime}(M - M_0)Y^\star / r(M - M_0)}{Y'(I - M)Y / r(I - M)} \\ &= \frac{\big((MY^\star)'(MY^\star) - (M_0Y^\star)'(M_0Y)\big)/r(M - M_0)}{\big((Y'Y - (MY)'(MY)\big)/r(I - M)} \\ &= \frac{\frac{1}{r}\left(\sum_{i=1}^r y_{1i} - \Delta\right)^2 + \frac{1}{s}\left(\sum_{i=1}^s y_{2i}\right)^2 - \frac{1}{r+s}\left(\sum_{i=1}^r (y_{1i} - \Delta) + \sum_{i=1}^s y_{2i}\right)}{\left(\sum_{i=1}^r (y_{1i}^2 - \overline{y}_1^2) + \sum_{i=1}^s (y_{2i}^2 - \overline{y}_2^2)\right)/(r + s - 2)}. \end{aligned}$$

Therefore, reject H_0 if $F > f(1 - \alpha, 1, r + s - 2)$.

Problem 3:

Redo the tests in Exercise 2.3 using the procedures of Section 3.2. Write down the models and explain the procedure. Hints: (a) Let A be a matrix of zeros, the generalized inverse of A, A^-, can be anything at all because $AA^-A = A$ for any choice of A^-. (b) There is no reason why X_0 cannot be a matrix of zeros. **Exercise 2.3:** Let $y_1, ..., y_n$ be independent $N(\mu, \sigma^2)$. Write a linear model for these data. Form an $\alpha = .01$ test for $H_0 \colon \mu = \mu_0$, where μ_0 is some known fixed number and form a 95% confidence interval for μ. How do these results compare with the usual analysis for one sample?

Solution: The full linear model $Y = X\beta + \epsilon$ can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

where $\epsilon \sim N(0, \sigma^2 I)$. Then, the reduced model is $Y = X\beta_0 + \epsilon$ and can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_0 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

where $\epsilon \sim N(0, \sigma^2 I)$, i.e. $X_0 = 0$. Then, we calculate the projection matrices

$$\begin{aligned} M &= X(X'X)^- X' = \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{n}} \end{bmatrix}_{n \times n} \\ M_0 &= X_0(X_0'X_0)^- X_0' = \begin{bmatrix} \mathbf{0} \end{bmatrix}_{n \times n} \end{aligned}$$

and therefore we have $M - M_0 = M$. Notice that $r(M - M_0) = r(M) = 1$ and $r(I - M) = n - r(M) = n - 1$. Then, we calculate

$$\begin{aligned} (Y - X\beta_0)(M - M_0)(Y - X\beta_0) &= (Y - X\beta_0)M(Y - X\beta_0) \\ &= (MY - MX\beta_0)'(MY - X\beta_0) \\ &= \begin{bmatrix} \overline{y} - \mu_0 & \cdots & \overline{y} - \mu_0 \end{bmatrix} \begin{bmatrix} \overline{y} - \mu_0 \\ \vdots \\ \overline{y} - \mu_0 \end{bmatrix} \\ &= n(\overline{y} - \mu_0)^2. \end{aligned}$$

and

$$Y'(I - M)Y = Y'Y - Y'MY = \sum_{i=1}^{n} y_i^2 - n\overline{y}^2.$$

Then, we obtain the test statistic

$$\begin{aligned} F &= \frac{(Y - X\beta_0)'(M - M_0)(Y - X\beta_0)/r(M - M_0)}{Y'(I - M)Y/r(I - M)} \\ &= \frac{n(\overline{y} - \mu_0)^2}{\left(\sum_{i=1}^{n} y_i^2 - n\overline{y}^2\right)/(n - 1)}. \end{aligned}$$

Therefore, with $\alpha = .01$, reject H_0 if $F > f(.99, 1, n - 1)$.

Problem 4:

Show that $\beta' X' M_{MP} X\beta = 0$ if and only if $\Lambda'\beta = 0$.

Solution: Let $\Lambda'\beta = 0$. Then, $P'X\beta = 0$ and so $P'MX\beta = 0$. Then, notice that

$$\beta' X' M_{MP} X\beta = \beta' X' MP(P'MP)^- P'MX\beta = 0.$$

Now, let $\beta' X' M_{MP} X\beta = 0$. Then, $(M_{MP}X\beta)'(M_{MP}X\beta) = 0$, which implies that $M_{MP}X\beta = 0$. Then, $P'M_{MP}X\beta = 0$. Then, we have

$$\begin{aligned}
& P'M_{MP}X\beta = 0 \\
\Longrightarrow\quad & P'MP(P'MP)^{-}(MP)'X\beta = 0 \\
\Longrightarrow\quad & \big[(P'M)(MP)(P'MP)^{-}(MP)'\big]X\beta = 0 \\
\Longrightarrow\quad & \big[(MP)(P'MP)^{-}(MP)'(MP)\big]'X\beta = 0 \\
\Longrightarrow\quad & P'MX\beta = 0 \\
\Longrightarrow\quad & P'X\beta = 0.
\end{aligned}$$

Therefore, $\Lambda'\beta = 0$.

Problem 5:

Consider a set of seemingly unrelated regression equations

$$Y_i = X_i\beta_i + e_i, \qquad e_i \sim N(0, \sigma^2 I),$$

$i = 1, ..., r$, where X_i is an $n_i \times p$ matrix and the e_is are independent. Find the test for $H_0\colon \beta_1 = ... = \beta_r$.

Solution: Notice the full model $Y = X\beta + e$ can be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{bmatrix} = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ 0 & X_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & X_r \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_r \end{bmatrix}.$$

Then, the reduced model $Y = X_0\gamma + e$ can be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix} \big[\beta_1\big] + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_r \end{bmatrix}.$$

Then, calculating $M = X(X'X)^{-}X'$ and $M_0 = X_0(X_0'X_0)^{-}X_0'$, we obtain the statistic

$$F = \frac{Y'(M - M_0)Y/r(M - M_0)}{Y'(I - M)Y/r(I - M)}.$$

Then, reject H_0 if $F > f\big(1 - \alpha, r(M - M_0), r(I - M)\big)$.